

Symmetries and color symmetries of a family of tilings with a singular point

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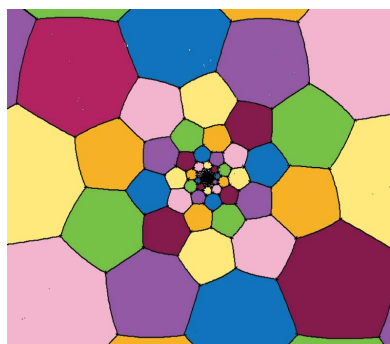
Tilings with a singular point are obtained by applying conformal maps on regular tilings of the Euclidean plane and their symmetries are determined. The resulting tilings are then symmetrically colored by applying the same conformal maps on colorings of regular tilings arising from sublattice colorings of the centers of the tiles. In addition, conditions are determined in order that the coloring of a tiling with singularity that is obtained in this manner is perfect.

1. Introduction and outline

A singularity of a tiling is a point P for which any circular disc centered at P intersects an infinite number of tiles. Hence, tilings with singular points are not locally finite and are classified among tilings that are not ‘well behaved’ (Grünbaum & Shephard, 1987). Geometric and topological properties of tilings with singularities have been investigated by Breen (1983), Nielsen (1990) and Sushida *et al.* (2012).

Symmetric colorings of tilings with a singular point were obtained by Lück (2010) by distorting certain colorings of regular Euclidean tilings. However, it was found that not all colorings of regular tilings could be transformed into colorings of tilings with a singular point, because some colorings were incompatible with the tiling’s rotational symmetry. Furthermore, it was surmised that there is a maximum number of colors for such colorings of a tiling with singularity. In Fig. 1(a), we have recreated a coloring of a tiling with a singularity from Fig. 6 of Lück (2010). Fig. 1(b) shows the coloring of a regular tiling by squares from which Fig. 1(a) was obtained. Other colorings of the square tiling also compatible with the rotational symmetry of the tiling with singularity were found to have 1, 2, 5, 10, 25 and 50 colors only.

In this contribution, we provide a mathematical basis for the observations made by Lück (2010). We begin by introducing in §2 the framework used in this paper, together with some known results. In §3 we discuss the conformal maps that are applied on regular Euclidean tilings which yield tilings with a singularity. In §4, a correspondence between the symmetry groups of the regular tilings and their images under the conformal maps is established. This allows us to determine the symmetry group of the tiling with singularity, which turns out to be isomorphic to a finite cyclic or finite dihedral group. In §5 we focus on the symmetry of colorings of tilings with a singularity. We use a method similar to that of Lück (2010) to obtain colorings of tilings with a singularity: conformal maps are applied on sublattice colorings of regular Euclidean tilings. However, applying a conformal map on a coloring of a regular tiling does not always yield a coloring of the corresponding tiling with singularity. Consequently, a compatibility condition



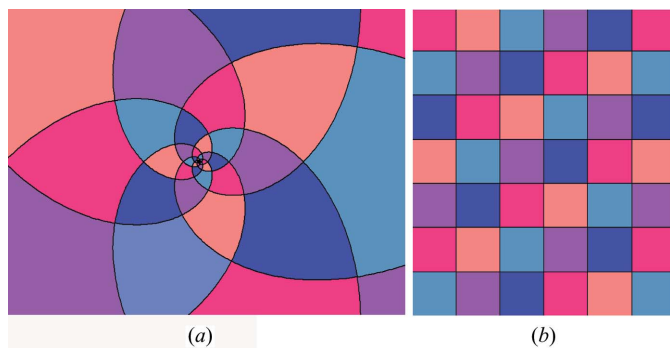


Figure 1
(a) Coloring of a tiling with a singularity and (b) corresponding coloring of the (4^4) tiling.

between a sublattice coloring of a regular tiling and a conformal map is established. Furthermore, for colorings compatible with a conformal map, we identify a necessary and sufficient condition so that a symmetry of the uncolored tiling with singularity is a color symmetry of the resulting coloring of the tiling. Lastly, we determine the maximum number of colors of a chirally perfect coloring of the tiling with a singularity obtained from compatible ideal colorings of the associated regular tiling. Several examples are presented in §6.

2. Preliminaries

A planar *tile* is any set T in the Euclidean plane that is the closure of its interior. Here, tiles are always bounded and, consequently, compact. A *tiling* of \mathbb{R}^2 is a set of tiles $\mathcal{T} = \{T_i\}_{i \in \mathbb{N}}$, such that $\cup_{i \in \mathbb{N}} T_i = \mathbb{R}^2$ and $\text{int}(T_i) \cap \text{int}(T_j) = \emptyset$ for all $i \neq j$. That is, \mathcal{T} is both a packing and a covering of \mathbb{R}^2 .

A tiling \mathcal{T} of \mathbb{R}^2 is said to be *locally finite* if, for any given point $x \in \mathbb{R}^2$, every circular disc centered at x intersects only a finite number of tiles of \mathcal{T} (Grünbaum & Shephard, 1987; Baake & Grimm, 2013). Hence, if \mathcal{T} is not locally finite, then there exists at least one point $x_0 \in \mathbb{R}^2$ such that, for all $r > 0$, $B(x_0, r) \cap T \neq \emptyset$ for an infinite number of $T \in \mathcal{T}$. Such a point is called a *singular point* of the tiling.

We are primarily concerned with symmetries of tilings with a singularity at the origin that are obtained by applying conformal maps on the three types of regular tilings of \mathbb{R}^2 : the (4^4) tiling (regular tiling by squares), the (6^3) tiling (regular tiling by hexagons) and the (3^6) tiling (regular tiling by triangles). Furthermore, we determine the color symmetries of colorings of tilings with a singularity obtained from certain colorings of regular tilings. To this end, the following definitions taken from Grünbaum & Shephard (1987) are needed.

Let X be a set of points or a set of tiles in \mathbb{R}^2 , with symmetry group $S(X)$. We refer to the subgroup of $S(X)$ containing all direct symmetries of X as the *rotation group* of X , and denote it by $R(X)$. Meanwhile, the subgroup of $S(X)$ containing all translation symmetries of X is called the *translation group* of X , and is written $T(X)$.

A *coloring* of X is a surjective map $C_X : X \rightarrow K$, where K is a finite set of m colors. Equivalently, a coloring C_X of X may be viewed as a partition $\{X_i\}_{i=1}^m$ of X . A *color symmetry* of C_X is an

element of $S(X)$ that permutes the colors of C_X . The set \mathcal{H}_{C_X} of all color symmetries of C_X , that is $\mathcal{H}_{C_X} = \{h \in S(X) \mid \exists \text{ permutation } \sigma_h \text{ on } K \text{ such that } \forall x \in X, C_X(h(x)) = \sigma_h(C_X(x))\}$, forms a subgroup of $S(X)$. We call this the *color symmetry group* of the coloring C_X . If $\mathcal{H}_{C_X} = S(X)$, then C_X is said to be a *perfect coloring* of X . If \mathcal{H}_{C_X} contains $R(X)$, then C_X is called a *chirally perfect coloring* of X (Rigby, 1998). To distinguish a perfect coloring from a chirally perfect coloring, we sometimes refer to a perfect coloring as a *fully perfect coloring*.

Of particular interest are *sublattice colorings* of square and hexagonal lattices (Moody & Patera, 1994; De las Peñas & Felix, 2007). A sublattice Γ of a lattice Γ_0 is a subgroup of finite index m in Γ_0 . We assign a unique color to each coset of Γ to obtain a coloring of Γ_0 . That is, the sublattice coloring of Γ_0 induced by Γ is given by $C_{\Gamma_0} = \{X_i\}_{i=1}^m$, where X_i , $i \in \{1, \dots, m\}$, is a coset of Γ . Since the number of colors in the coloring of Γ_0 induced by Γ is the index of Γ in Γ_0 , we sometimes refer to the number of colors as the *color index* of the coloring. Furthermore, the translation symmetries of Γ_0 are always contained in $\mathcal{H}_{C_{\Gamma_0}}$ because they map cosets of Γ to cosets of Γ [see Theorem 1 of De las Peñas & Felix (2007)].

Finally, we describe the setting that we will use throughout the paper.

We identify the Euclidean plane \mathbb{R}^2 with the complex plane \mathbb{C} . If $z \in \mathbb{C}$, we denote the modulus of z by $|z|$, and its conjugate by \bar{z} . We use i and ω to represent $(-1)^{1/2}$ and $\exp(2\pi i/3)$, respectively. Furthermore, we associate the square lattice with the ring $\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$ of Gaussian integers, while the hexagonal lattice is identified with the ring $\mathbb{Z}[\omega] = \{a + b\omega \mid a, b \in \mathbb{Z}\}$ of Eisenstein integers. It is known that if $\xi = i$ or $\xi = \omega$, then $\mathbb{Z}[\xi]$ is a principal ideal domain. Also, given $\beta \in \mathbb{Z}[\xi]$, the ideal (β) generated by β is of index $|\beta|^2$ in $\mathbb{Z}[\xi]$.

Let \mathcal{T} be a regular tiling, P be the set of centers of tiles of \mathcal{T} , and Λ be the orbit of the origin under the action of $T(\mathcal{T})$. We superimpose the complex plane over \mathcal{T} in the following manner:

\mathcal{T} is the (4^4) tiling: the set P is a square lattice. We superimpose \mathbb{C} over \mathcal{T} so that P coincides with the points of $\mathbb{Z}[i]$. Note that $\Lambda = \mathbb{Z}[i]$.

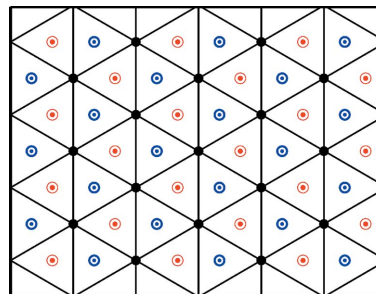


Figure 2
 (3^6) Tiling: the black points represent the ideal $P_0 = (2 + \omega)$, the blue points (thick outer shell) represent $P_1 = -1 + (2 + \omega)$, and the red points (thin outer shell) represent $P_2 = 1 + (2 + \omega)$.

\mathcal{T} is the (6^3) tiling: the set P forms a hexagonal lattice. We view \mathcal{T} in \mathbb{C} in such a way that P coincides with $\mathbb{Z}[\omega]$, from which follows $\Lambda = \mathbb{Z}[\omega]$.

\mathcal{T} is the (3^6) tiling: the set P is not a lattice, but nonetheless forms a crystallographic point packing (Conway & Sloane, 1999; Baake & Grimm, 2013). Superimpose \mathbb{C} over \mathcal{T} so that the set P_0 of vertices of \mathcal{T} coincides with the points of the ideal $(2 + \omega)$ of $\mathbb{Z}[\omega]$. The two other cosets of P_0 in $\mathbb{Z}[\omega]$ are $P_1 = -1 + (2 + \omega)$ and $P_2 = 1 + (2 + \omega)$. Note that $P = P_1 \cup P_2$, while $\Lambda = P_0$. This is illustrated in Fig. 2.

In all three cases the symmetry group of \mathcal{T} is $S(\mathcal{T}) = \langle h_1, h_2, h_3, h_4 \rangle$, where h_1 is a rotation, h_2 is a reflection, and h_3 and h_4 are translations. Table 1 shows these symmetries as complex maps for the three regular tilings.

Generating functions for the number of colors in perfect colorings of $\mathbb{Z}[i]$ and $\mathbb{Z}[\omega]$ are known (Baake & Grimm, 2004). On the other hand, the following result from Bugarin *et al.* (2013) provides necessary and sufficient conditions for a sublattice coloring of $\mathbb{Z}[i]$ or of $\mathbb{Z}[\omega]$ to be chirally or fully perfect.

Theorem 1. Let $\xi \in \{i, \omega\}$ and $C_{\mathbb{Z}[\xi]}$ be a coloring induced by the sublattice Γ . Then the coloring $C_{\mathbb{Z}[\xi]}$ is chirally perfect if and only if Γ is an ideal of $\mathbb{Z}[\xi]$. In addition, if $\beta \in \mathbb{Z}[\xi]$, then the coloring $C_{\mathbb{Z}[\xi]}$ induced by (β) is perfect if and only if $\beta = \pm \xi^j \bar{\beta}$ for some $j \in \mathbb{Z}$.

Theorem 1 states that a sublattice coloring of $\mathbb{Z}[i]$ or of $\mathbb{Z}[\omega]$ is chirally perfect if and only if the sublattice that induces the coloring is an ideal of the respective lattice. We refer to such colorings as *ideal colorings* of the lattice. Furthermore, this ideal coloring is perfect if its generator is *balanced*, that is, the generator and its conjugate are associates (Washington, 1996).

3. Family of tilings with a singularity via conformal mappings

We consider the map $\varphi_\alpha : \mathbb{C} \rightarrow \mathbb{C}^* := \mathbb{C} \setminus \{0\}$ defined by

$$\varphi_\alpha(z) = \exp\left(\frac{2\pi iz}{\alpha}\right) = \exp\left(\frac{2\pi i}{|\alpha|^2} \bar{\alpha} z\right) \quad (1)$$

where $\alpha \in \mathbb{C}^*$. The map φ_α is conformal, since it is the composition of the exponential map and a linear map which are both conformal (Brown & Churchill, 2008).

The following lemma establishes which points of \mathbb{C} have the same image under φ_α .

Lemma 2. Let $\varphi_\alpha(z) = \exp(2\pi iz/\alpha)$, where $\alpha \in \mathbb{C}^*$. Given $z_1, z_2 \in \mathbb{C}$, then $\varphi_\alpha(z_1) = \varphi_\alpha(z_2)$ if and only if $z_2 = z_1 + k\alpha$ for some $k \in \mathbb{Z}$.

We now apply φ_α on a regular tiling \mathcal{T} to obtain a tiling with a singularity at the origin. For this, we need to find conditions on α that guarantee that $\mathcal{S}_\alpha := \{\varphi_\alpha(T) \mid T \in \mathcal{T}\}$ is a tiling of \mathbb{C}^* .

Table 1
Symmetries of regular tilings.

Tiling	(4^4)	(6^3)	(3^6)
$h_1(z) =$	iz	$(1 + \omega)z$	$(1 + \omega)z$
$h_2(z) =$	\bar{z}	\bar{z}	\bar{z}
$h_3(z) =$	$z + 1$	$z + 1$	$z + (2 + \omega)$
$h_4(z) =$	$z + i$	$z + \omega$	$z + (1 - \omega)$
$S(\mathcal{T})$ of type	$p4m$ or $*442$	$p6m$ or $*632$	$p6m$ or $*632$

Theorem 3. Let \mathcal{T} be a regular tiling, Λ be the orbit of the origin under the action of $T(\mathcal{T})$ and $\varphi_\alpha(z) = \exp(2\pi iz/\alpha)$. If $\alpha \in \Lambda$ and

$$|\alpha| > \max\{|z_1 - z_2| \mid z_1, z_2 \in \partial T, T \in \mathcal{T}\}$$

(where ∂T denotes the boundary of tile T), then

$$\mathcal{S}_\alpha = \{\varphi_\alpha(T) \mid T \in \mathcal{T}\}$$

is a tiling of \mathbb{C}^* .

Proof. Let $T \in \mathcal{T}$. Clearly, $\varphi_\alpha(\partial T)$ is a closed curve. Suppose $z_1, z_2 \in \partial T$ such that $\varphi_\alpha(z_1) = \varphi_\alpha(z_2)$. Then $z_1 = z_2 + k\alpha$ for some $k \in \mathbb{Z}$ by Lemma 2. We have $|\alpha| > |z_1 - z_2| = |k||\alpha|$ which implies that $k = 0$ and $z_1 = z_2$. Thus, φ_α maps ∂T one-to-one and onto $\varphi_\alpha(\partial T)$. This means that $\varphi_\alpha(\partial T)$ is simple, and it follows from ch. 3 of Bieberbach (1953) that $\text{int}(T)$ is also mapped one-to-one and onto $\text{int}(\varphi_\alpha(\partial T))$. This implies that $\varphi_\alpha(T)$ is a tile.

Since \mathcal{T} is a covering of \mathbb{C} and φ_α is onto, \mathcal{S}_α is a covering of \mathbb{C}^* . All that remains is to show that \mathcal{S}_α is a packing of \mathbb{C}^* . Indeed, let $T_1, T_2 \in \mathcal{T}$ such that $\text{int}(\varphi_\alpha(T_1)) \cap \text{int}(\varphi_\alpha(T_2)) \neq \emptyset$. Then there exist $z_1 \in \text{int}(T_1)$ and $z_2 \in \text{int}(T_2)$ such that $z_2 = z_1 + k\alpha$ for some $k \in \mathbb{Z}$ by Lemma 2. Since $\alpha \in \Lambda$, we have $z_2 \in \text{int}(T_1 + k\alpha)$. Hence, $T_2 = T_1 + k\alpha$ because \mathcal{T} is a packing of \mathbb{C} , and thus, $\varphi_\alpha(T_1) = \varphi_\alpha(T_2)$. \square

Henceforth, it shall be assumed that α satisfies the required conditions stated in Theorem 3 so that \mathcal{S}_α is a tiling of \mathbb{C}^* . We say that such an α is *admissible*. We make the following observations about admissible values of α :

(i) Since $\alpha \in \Lambda$, there exist unique integers L and R such that $\alpha = L + Ri$, $\alpha = L + R\omega$ and $\alpha = (L + R\omega)(2 + \omega) = (2L - R) + (L + R)\omega$ for the (4^4) , (6^3) and (3^6) tilings, respectively. We now say that L and R are the *integers that determine α* .

(ii) Since $|\alpha| > \max\{|z_1 - z_2| \mid z_1, z_2 \in \partial T, T \in \mathcal{T}\}$, $|\alpha|$ is greater than $2^{1/2}$, $[2(3^{1/2})]/3$ and $3^{1/2}$ for the (4^4) , (6^3) and (3^6) tilings, respectively.

The following proposition states that if α is an admissible value, then replacing α by any of its associates in equation (1) yields the same tiling \mathcal{S}_α . Thus, to exhaust all possible \mathcal{S}_α , it is enough to consider one representative from each associate class of $\mathbb{Z}[\xi]$.

Proposition 4. Let $\varphi_{\alpha_1}(z) = \exp(2\pi iz/\alpha_1)$, $\varphi_{\alpha_2}(z) = \exp(2\pi iz/\alpha_2)$ and $\xi \in \{i, \omega\}$. If α_1 and α_2 are associates in $\mathbb{Z}[\xi]$, then $\mathcal{S}_{\alpha_1} = \mathcal{S}_{\alpha_2}$.

Table 2
Three classes of tilings with a singularity at the origin and the curves that bound their tiles.

Class	\mathcal{T}	Images of edges of \mathcal{T} under φ_α
1	(4 ⁴)	Two sets of spirals that contain $ L $ and $ R $ spirals, spirals in one set are positively oriented while spirals in the other set are negatively oriented
	(6 ³)	Arcs of three sets of spirals that contain $ 2L - R $, $ 2R - L $ and $ L + R $ spirals, spirals in two sets go in the same direction while spirals in the third set go in the opposite direction
	(3 ⁶)	Three sets of spirals that contain $ L $, $ R $ and $ L - R $ spirals, spirals in two sets go in the same direction while spirals in the third set go in the opposite direction
2	(4 ⁴)	$ L $ positively and $ L $ negatively oriented spirals
	(6 ³)	Arcs of $ 3L $ positively and $ 3L $ negatively oriented spirals, arcs of circles centered at the origin
	(3 ⁶)	$ L $ positively and $ L $ negatively oriented spirals, circles centered at the origin
3	(4 ⁴)	$ L $ rays emanating from the origin, circles centered at the origin
	(6 ³)	Arcs of $ L $ positively and $ L $ negatively oriented spirals, segments of $ 2L $ rays emanating from the origin
	(3 ⁶)	$ L/3 $ positively and $ L/3 $ negatively oriented spirals, $ 2L/3 $ rays emanating from the origin

Proof. Let $\xi = i$ if \mathcal{T} is the (4⁴) tiling, and $\xi = \omega$ if \mathcal{T} is the (6³) or (3⁶) tiling. If α_1 and α_2 are associates, then $\alpha_2 = \varepsilon\alpha_1$ for some unit $\varepsilon \in \mathbb{Z}[\xi]$. Now $f(z) = \varepsilon z$ is a symmetry of \mathcal{T} , so $f(T) = \varepsilon T \in \mathcal{T}$ whenever $T \in \mathcal{T}$. The conclusion then follows since $\varphi_{\alpha_1}(T) = \varphi_{\alpha_2}(\varepsilon T)$. \square

We now describe the tilings \mathcal{S}_α . Clearly, the edges of \mathcal{T} are contained in lines of \mathbb{C} . Thus, the map φ_α sends an edge of \mathcal{T} to one of the following: an arc of a logarithmic spiral with the asymptotic point at the origin, an arc of a circle centered at the origin, and a segment of an open-ended ray emanating from the origin. This is the reason why \mathcal{S}_α ends up having a singular point at the origin.

The edges of \mathcal{T} can be partitioned into equivalence classes, where two edges belong to the same equivalence class if they are parallel. There are two equivalence classes of parallel edges for the (4⁴) tiling, and three for the (6³) and (3⁶) tilings. It

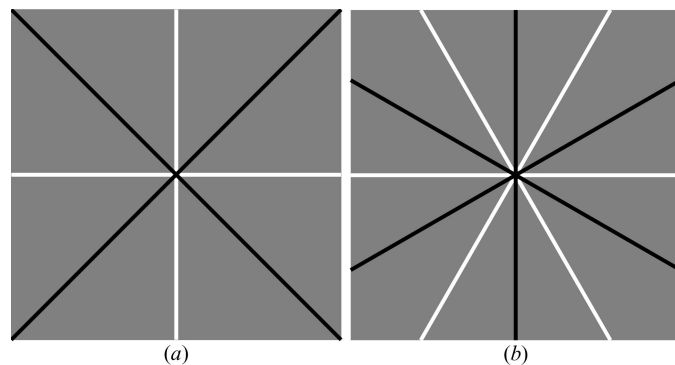


Figure 3
(a) (4⁴) Tiling, (b) (6³) and (3⁶) tilings. The singular tiling \mathcal{S}_α falls under Class 1 if α lies in the gray area, under Class 2 if α lies on one of the black lines, and under Class 3 if α lies on one of the white lines.

is easy to verify that the images of edges that belong to the same equivalence class are of the same type.

The image under φ_α of an equivalence class of parallel edges in \mathcal{T} depends on the prime decomposition of α in $\mathbb{Z}[\xi]$, where $\xi = i$ if \mathcal{T} is the (4⁴) tiling and $\xi = \omega$ if \mathcal{T} is the (6³) or (3⁶) tiling. Hence, we categorize all possible \mathcal{S}_α into three classes.

Class 1 tilings: α is not balanced.

Class 2 tilings: α is balanced and the factorization of α contains an odd power of a factor of the ramified prime in $\mathbb{Z}[\xi]$, that is, we can write $\alpha = \varepsilon(L - L\xi)$ for some unit ε in $\mathbb{Z}[\xi]$ and $L \in \mathbb{Z}$.

Class 3 tilings: α is an integer multiple of a unit in $\mathbb{Z}[\xi]$, that is, $\alpha = \varepsilon L$ for some unit ε in $\mathbb{Z}[\xi]$ and $L \in \mathbb{Z}$.

Fig. 3 shows where α lies in the complex plane for each of the above classes. We also summarize the curves that bound the tiles of \mathcal{S}_α for each class in Table 2.

Let $[\ell]$ be an equivalence class of edges of \mathcal{T} . Observe from Table 2 that if $\varphi_\alpha([\ell])$ is a set of (arcs of) logarithmic spirals or a set of (segments of) rays, then it contains only a finite number of elements. In addition, as we will see in the next section, the spirals and rays are spread out symmetrically about the origin. On the other hand, if $\varphi_\alpha([\ell])$ is a set of (arcs

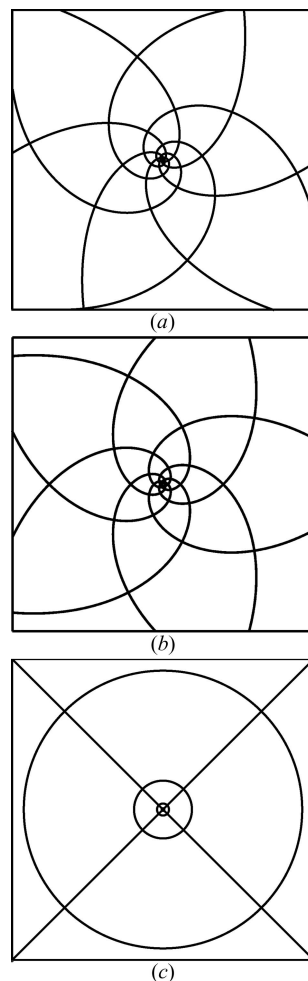


Figure 4
(a) Class 1 ($\alpha = 4 + 6i$), (b) Class 2 ($\alpha = -5 + 5i$), (c) Class 3 ($\alpha = 4$). Tilings with singularity obtained by applying the conformal map φ_α on the (4⁴) tiling.

of) circles, then it is countably infinite. Here, the radii of the circles increase geometrically. Fig. 4 shows an example from each class for the (4^4) tiling.

4. Symmetries of \mathcal{S}_α

Since \mathcal{S}_α has only one singular point, a symmetry of \mathcal{S}_α must fix that point. This implies that $S(\mathcal{S}_\alpha)$ is a group of isometries that fix the origin and, consequently, it must be isomorphic to some finite cyclic group C_n or finite dihedral group D_n . Thus, in order to identify $S(\mathcal{S}_\alpha)$, we only need to find a generator of $R(\mathcal{S}_\alpha)$ and to determine whether any reflection symmetry leaves \mathcal{S}_α invariant. For this, we turn to the symmetries of \mathcal{T} .

Lemma 5. Let \mathcal{T} be a regular tiling, $\varphi_\alpha(z) = \exp(2\pi iz/\alpha)$ with α admissible, and $\mathcal{S}_\alpha = \{\varphi_\alpha(T) \mid T \in \mathcal{T}\}$. If g is an isometry of \mathbb{C}^* such that there exists a symmetry $f \in S(\mathcal{T})$ for which $g\varphi_\alpha = \varphi_\alpha f$, then $g \in S(\mathcal{S}_\alpha)$.

Proof. We have

$$g(\mathcal{S}_\alpha) = g(\varphi_\alpha(\mathcal{T})) = \varphi_\alpha(f(\mathcal{T})) = \varphi_\alpha(\mathcal{T}) = \mathcal{S}_\alpha.$$

□

If $g \in S(\mathcal{S}_\alpha)$ and $f \in S(\mathcal{T})$ for which the diagram in Fig. 5 commutes, then we say that f corresponds to g . Note that an $f \in S(\mathcal{T})$ that corresponds to $g \in S(\mathcal{S}_\alpha)$ is not unique. Indeed, if $h = f + k\alpha \in S(\mathcal{T})$ for some $k \in \mathbb{Z}$, then h also corresponds to g . We now use Lemma 5 to find the symmetry group of \mathcal{S}_α .

Theorem 6. Let \mathcal{T} be a regular tiling, $\varphi_\alpha(z) = \exp(2\pi iz/\alpha)$ with α admissible, and $\mathcal{S}_\alpha = \{\varphi_\alpha(T) \mid T \in \mathcal{T}\}$. Suppose $n = \gcd(L, R)$, where L and R are the integers that determine α . Define $g_1(z) = \exp(2\pi i/n)z$ and $g_2(z) = \bar{z}$.

- (a) If α is not balanced, then $S(\mathcal{S}_\alpha) = \langle g_1 \rangle \cong C_n$.
- (b) If α is balanced, then $S(\mathcal{S}_\alpha) = \langle g_1, g_2 \rangle \cong D_n$.

Proof. Let $m \in \mathbb{N}$ such that $m \mid \alpha$. Then $g(z) = \exp(2\pi i/m)z \in S(\mathcal{S}_\alpha)$, and a symmetry of \mathcal{T} that corresponds to g is the translation given by $f(z) = z + \alpha/m$. Hence, the rotation group $R(\mathcal{S}_\alpha)$ is generated by g_1 .

If α is not balanced, then \mathcal{S}_α falls under Class 1. Since the number of positively oriented spirals and the number of negatively oriented spirals are not the same (see Table 2), \mathcal{S}_α cannot have a reflection symmetry.

If α is balanced, then g_2 leaves \mathcal{S}_α invariant. Indeed, a symmetry of \mathcal{T} that corresponds to g_2 is the reflection f_2 given

by $f_2(z) = \varepsilon \bar{z}$ for some unit ε of $\mathbb{Z}[\xi]$ [where $\xi = i$ if \mathcal{T} is the (4^4) tiling and $\xi = \omega$ if \mathcal{T} is the (6^3) or (3^6) tiling]. □

Not only does Theorem 6 give us the symmetry group of \mathcal{S}_α , but it also tells us that for any positive integer n , it is possible to construct a tiling \mathcal{S}_α that has rotation symmetry of order n about the origin by choosing a suitable α . Moreover, we obtain that for every $g \in S(\mathcal{S}_\alpha)$, there exists $f \in S(\mathcal{T})$ such that f corresponds to g .

In fact, the tilings \mathcal{S}_α are not only symmetrical, but also self-similar with a scaling factor that is some power of e . This has also been observed by Lück (2010), where some tilings with a singularity even have scaling factors related to quasiperiodic patterns such as the golden ratio τ . However, this aspect of \mathcal{S}_α will not be discussed here but will be published elsewhere.

There is a one-to-one correspondence λ between a regular tiling \mathcal{T} of \mathbb{R}^2 and the set P of centers of tiles of \mathcal{T} . By inspection, we see that \mathcal{T} and P have the same symmetry group. Furthermore, $\lambda(f(T)) = f(\lambda(T))$ holds for all $f \in S(\mathcal{T})$ and $T \in \mathcal{T}$. That is, the (group) action of $S(\mathcal{T})$ on \mathcal{T} is equivalent to its action on P . This means that every coloring of \mathcal{T} may be viewed as a coloring of P , and both colorings have the same color symmetry group. We now proceed to show a similar relationship between \mathcal{S}_α and $Q_\alpha := \{\varphi_\alpha(t) \mid t \in P\}$.

Lemma 7. Let \mathcal{T} be a regular tiling, P be the set of centers of \mathcal{T} and $\varphi_\alpha(z) = \exp(2\pi iz/\alpha)$ with α admissible. Suppose $\mathcal{S}_\alpha = \{\varphi_\alpha(T) \mid T \in \mathcal{T}\}$, $Q_\alpha = \{\varphi_\alpha(t) \mid t \in P\}$ and $\lambda : \mathcal{T} \rightarrow P$ is the map that sends a tile of \mathcal{T} to its center. Then the map $\sigma : \mathcal{S}_\alpha \rightarrow Q_\alpha$ that sends the image of a tile under φ_α to the image of its center under φ_α , that is, $(\sigma\varphi_\alpha)(T) = (\varphi_\alpha\lambda)(T)$ for $T \in \mathcal{T}$, defines a one-to-one correspondence between \mathcal{S}_α and Q_α .

Proof. Suppose that $T_1, T_2 \in \mathcal{T}$ such that $\varphi_\alpha(T_1) = \varphi_\alpha(T_2)$. Then there exists $t' \in T_1$ such that $\varphi_\alpha(t') = \varphi_\alpha(\lambda(T_2))$. By Lemma 2, $t' = \lambda(T_2) + k\alpha$ for some $k \in \mathbb{Z}$. Since $\lambda(T_2) \in P$ and $\alpha \in \Lambda$, it follows that $t' = \lambda(T_1)$. Consequently, σ is well defined.

Now assume that $\sigma(\varphi_\alpha(T_1)) = \sigma(\varphi_\alpha(T_2))$. Then $\varphi_\alpha(\lambda(T_1)) = \varphi_\alpha(\lambda(T_2))$. This implies that $\lambda(T_1) = \lambda(T_2) + k\alpha$ for some $k \in \mathbb{Z}$. Since $\alpha \in \Lambda$, $T_1 = T_2 + k\alpha$ and we obtain $\varphi_\alpha(T_1) = \varphi_\alpha(T_2)$. Thus, σ is injective. Clearly, σ is surjective since λ is surjective. □

Lemma 7 gives us the commutative diagram in Fig. 6.

Theorem 8. Let P be the set of centers of a regular tiling \mathcal{T} . Suppose $\varphi_\alpha(z) = \exp(2\pi iz/\alpha)$ with α admissible, $\mathcal{S}_\alpha = \{\varphi_\alpha(T) \mid T \in \mathcal{T}\}$ and $Q_\alpha = \{\varphi_\alpha(t) \mid t \in P\}$. Then $S(\mathcal{S}_\alpha)$ is a subgroup of $S(Q_\alpha)$ and the actions of $S(\mathcal{S}_\alpha)$ on \mathcal{S}_α and on Q_α are equivalent.

Proof. Let $g \in S(\mathcal{S}_\alpha)$ and $\varphi_\alpha(T) \in \mathcal{S}_\alpha$. Suppose $f \in S(\mathcal{T})$ corresponds to g . Then $g(Q_\alpha) = \varphi_\alpha(f(P)) = \varphi_\alpha(P) = Q_\alpha$, and so $S(\mathcal{S}_\alpha) \subseteq S(Q_\alpha)$. Let σ be as defined in Lemma 7. Since the action of f on \mathcal{T} is equivalent to its action on P , we have

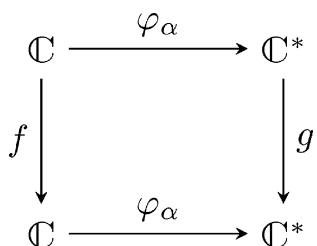


Figure 5 Commutative diagram involving the maps φ_α , g and f .

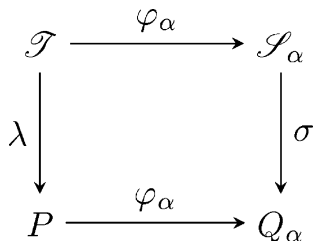


Figure 6
Commutative diagram involving the maps φ_α , σ and λ .

$$\begin{aligned}
 (g\sigma)(\varphi_\alpha(T)) &= g(\varphi_\alpha(\lambda(T))) = \varphi_\alpha(f(\lambda(T))) \\
 &= \varphi_\alpha(\lambda(f(T))) = \sigma(\varphi_\alpha(f(T))) = (\sigma g)(\varphi_\alpha(T)).
 \end{aligned}$$

This shows that the action of $S(\mathcal{S}_\alpha)$ on \mathcal{S}_α is equivalent to its action on Q_α . \square

Hence, there is a one-to-one correspondence between colorings of \mathcal{S}_α and colorings of Q_α via the map σ . Furthermore, elements of $S(\mathcal{S}_\alpha)$ that are color symmetries of a coloring of Q_α are also color symmetries of the associated coloring of \mathcal{S}_α .

An open question at this point is whether or not $S(\mathcal{S}_\alpha)$ is strictly contained in $S(Q_\alpha)$. Note that the points in Q_α are not necessarily the geometric centers of the tiles in \mathcal{S}_α , and so it might happen that Q_α is more symmetric than \mathcal{S}_α . Nonetheless, it suffices that $S(\mathcal{S}_\alpha)$ is a subgroup of $S(Q_\alpha)$ for the rest of the paper.

5. Color symmetries of \mathcal{S}_α

Our aim in this section is to come up with chirally and fully perfect colorings of the tiling $\mathcal{S}_\alpha = \varphi_\alpha(\mathcal{T})$ that have a singularity at the origin. In Lemma 7, we defined λ to be the bijection that sends a tile $T \in \mathcal{T}$ to its center $t \in P$, and σ to be the bijection that sends $\varphi_\alpha(T) \in \mathcal{S}_\alpha$ to $\varphi_\alpha(\lambda(T)) \in Q_\alpha$. If $\mathcal{C}_P = \{X_i\}_{i=1}^m$ is a coloring of P , then $\mathcal{C}_\mathcal{T} = \{\lambda^{-1}(X_i)\}_{i=1}^m$ gives a coloring of \mathcal{T} . Similarly, if $\mathcal{C}_{Q_\alpha} = \{Y_i\}_{i=1}^k$ is a coloring of Q_α , then $\mathcal{C}_{\mathcal{S}_\alpha} = \{\sigma^{-1}(Y_i)\}_{i=1}^k$ yields a coloring of \mathcal{S}_α . In addition, Theorem 8 assures us that if the elements of $R(\mathcal{S}_\alpha)$ [or $S(\mathcal{S}_\alpha)$] permute the colors of \mathcal{C}_{Q_α} , then $\mathcal{C}_{\mathcal{S}_\alpha}$ is a chirally (or fully) perfect coloring. All that remains is to obtain colorings of Q_α from colorings of P .

5.1. Ideal colorings of P

We start by identifying the chirally and fully perfect colorings of the set P of centers of the regular tiling \mathcal{T} . If \mathcal{T} is the (4^4) or the (6^3) tiling, then $P = \mathbb{Z}[i]$ or $P = \mathbb{Z}[\omega]$. Hence, Theorem 1 applies and we already have the chirally and fully perfect colorings of P .

In the case where \mathcal{T} is the (3^6) tiling, P is not a lattice. Nonetheless, a coloring of P may still be obtained from colorings of the lattice $\mathbb{Z}[\omega]$. Indeed, let $\mathcal{C}_{\mathbb{Z}[\omega]} = \{X_i\}_{i=1}^m$ be the coloring of $\mathbb{Z}[\omega]$ induced by a sublattice Γ of $\mathbb{Z}[\omega]$. Since $P \subset \mathbb{Z}[\omega]$, we obtain a coloring of P from $\mathcal{C}_{\mathbb{Z}[\omega]}$ given by

$$\mathcal{C}_P = \{X_i \cap P \mid X_i \cap P \neq \emptyset, \quad 1 \leq i \leq m\}.$$

When needed, we re-index so that $\mathcal{C}_P = \{X_i \cap P\}_{i=1}^{m'}$ for some $m' \leq m$. In fact, m' is either m or $(2/3)m$, as shown in the next lemma.

Lemma 9. Let Γ be a sublattice of index m in $\mathbb{Z}[\omega]$, and $\mathcal{C}_{\mathbb{Z}[\omega]} = \{X_i\}_{i=1}^m$ be the coloring of $\mathbb{Z}[\omega]$ induced by Γ .

(a) If Γ is a sublattice of the ideal $P_0 = (2 + \omega)$, then the coloring of $P = \mathbb{Z}[\omega] \setminus P_0$ obtained from $\mathcal{C}_{\mathbb{Z}[\omega]}$ is $\mathcal{C}_P = \{X_i\}_{i=1}^{m'}$, where $m' = (2/3)m$.

(b) If Γ is not a sublattice of P_0 , then the coloring of P obtained from $\mathcal{C}_{\mathbb{Z}[\omega]}$ is

$$\mathcal{C}_P = \{X_i \cap P\}_{i=1}^m.$$

Proof. Suppose Γ is a sublattice of P_0 and $X_i \in \mathcal{C}_{\mathbb{Z}[\omega]}$. If X_i contains an element of P_0 , then $X_i \subseteq P_0$ and consequently $X_i \cap P = \emptyset$. Otherwise, $X_i \cap P = X_i$. Since $m = [\mathbb{Z}[\omega] : P_0][P_0 : \Gamma] = 3[P_0 : \Gamma]$, the number of cosets of Γ that lie entirely in P is $m - [P_0 : \Gamma] = (2/3)m$. Therefore, $\mathcal{C}_P = \{X_i\}_{i=1}^{m'}$, where $m' = (2/3)m$.

If Γ is not a sublattice of P_0 , then every $X_i \in \mathcal{C}_{\mathbb{Z}[\omega]}$ must contain an element which is not in P_0 . Hence, $X_i \cap P \neq \emptyset$ for all $i \in \{1, \dots, m\}$, which yields the claim. \square

We now establish a relationship between perfect colorings of $\mathbb{Z}[\omega]$ and perfect colorings of P . Observe that the generators h_1, h_2, h_3 and h_4 of $S(\mathcal{T})$ (see Table 1) permute $P_0, P_1 = -1 + (2 + \omega)$ and $P_2 = 1 + (2 + \omega)$. In particular, $h_1(P_0) = P_0, h_1(P_1) = P_2$ and $h_1(P_2) = P_1$, while h_2, h_3 and h_4 leave each coset invariant. We use this fact in the proofs of Lemma 10 and Theorem 11.

Lemma 10. Let Γ be a sublattice of $\mathbb{Z}[\omega]$ such that $[\mathbb{Z}[\omega] : \Gamma] = m$. Suppose $\mathcal{C}_{\mathbb{Z}[\omega]}$ is the coloring of $\mathbb{Z}[\omega]$ induced by Γ and \mathcal{C}_P is the coloring of P obtained from $\mathcal{C}_{\mathbb{Z}[\omega]}$. If $f \in S(\mathcal{T}) \cap O(2)$ is a color symmetry of \mathcal{C}_P , then f is a color symmetry of $\mathcal{C}_{\mathbb{Z}[\omega]}$.

Proof. Let $f \in S(\mathcal{T}) \cap O(2)$ such that $f(\Gamma) = \Gamma$. Since $f \in S(\mathbb{Z}[\omega])$, we have that $f(t + \Gamma) = f(t) + \Gamma$ is a coset of Γ for every coset $t + \Gamma$ of Γ . This implies that f permutes the cosets of Γ . Thus, to prove that f is a color symmetry of $\mathcal{C}_{\mathbb{Z}[\omega]}$, it suffices to show that f leaves Γ invariant.

Let $\mathcal{C}_{\mathbb{Z}[\omega]} = \{X_i\}_{i=1}^m$. First, we consider the case when Γ is a sublattice of P_0 . Then $\mathcal{C}_P = \{X_i\}_{i=1}^{m'}$. By assumption, given $X_i \in \mathcal{C}_P, f(X_i) = X_j$ for some $j \in \{1, \dots, m'\}$. If we write $X_i = t_i + \Gamma$ and $X_j = t_j + \Gamma$, then for all $\ell \in \Gamma, f(t_i + \ell) \in t_j + \Gamma$. This implies that $f(\ell) \in (t_j - f(t_i)) + \Gamma = \Gamma$. Hence, Γ contains $f(\Gamma)$ and so $f(\Gamma) = \Gamma$ because f is an isometry.

Now suppose Γ is not a sublattice of P_0 . We have $\mathcal{C}_P = \{X_i \cap P\}_{i=1}^m$. Let $t_0 \in \Gamma \cap P_0$. Given $t \in \Gamma \cap P$, we must have $t_0 + t \in P$ because P_0 is a sublattice of $\mathbb{Z}[\omega] = P_0 \cup P$. Hence, $t_0 + t \in \Gamma \cap P$. Since f permutes the colors of $\mathcal{C}_P, f(t)$ and $f(t_0 + t)$ must belong to the same coset of Γ . We get $f(t_0) = f(t_0 + t) - f(t) \in \Gamma$. Moreover, we have $f(t_0) \in P_0$. Thus, $f(\Gamma \cap P_0) \subseteq \Gamma \cap P_0$, and it follows that $f(\Gamma \cap P_0) =$

$\Gamma \cap P_0$. We obtain $f(\Gamma) = (\Gamma \cap P_0) \cup (X_j \cap P)$, for some $j \in \{1, \dots, m\}$. However, $f(\Gamma)$ must be a sublattice of $\mathbb{Z}[\omega]$, so $X_j = \Gamma$ and $f(\Gamma) = \Gamma$. \square

Theorem 11. Let Γ be a sublattice of index m in $\mathbb{Z}[\omega]$. Then the coloring $\mathcal{C}_{\mathbb{Z}[\omega]}$ of $\mathbb{Z}[\omega]$ induced by Γ is chirally (respectively, fully) perfect if and only if the coloring \mathcal{C}_P of P obtained from $\mathcal{C}_{\mathbb{Z}[\omega]}$ is chirally (respectively, fully) perfect.

Proof. If h_1 permutes (and h_2 permutes) the colors of \mathcal{C}_P , then Lemma 10 guarantees that h_1 permutes (and h_2 permutes) the colors of $\mathcal{C}_{\mathbb{Z}[\omega]}$. Since h_3 and h_4 are translation symmetries of $\mathbb{Z}[\omega]$, they must be color symmetries of $\mathcal{C}_{\mathbb{Z}[\omega]}$. This proves the backward direction.

Suppose $\mathcal{C}_{\mathbb{Z}[\omega]} = \{X_i\}_{i=1}^m$ is chirally (fully) perfect. Then given $X_i \in \mathcal{C}_{\mathbb{Z}[\omega]}$ for which $X_i \cap P \neq \emptyset$, $h_1(X_i) = X_j$ for some $j \in \{1, \dots, m\}$. We then have $h_1(X_i \cap P) = X_j \cap P \neq \emptyset$. Thus, h_1 permutes the colors of \mathcal{C}_P . In a similar manner, we can show that h_3, h_4 (and h_2) are also color symmetries of \mathcal{C}_P . \square

Hence, even if \mathcal{T} is the (3^6) tiling, ideal colorings of $\mathbb{Z}[\omega]$ give rise to chirally or fully perfect colorings of P . Conversely, if a coloring of P obtained from a sublattice coloring of $\mathbb{Z}[\omega]$ is chirally or fully perfect, then the coloring of $\mathbb{Z}[\omega]$ must be an ideal coloring.

5.2. Compatibility between sublattice colorings of P and conformal maps φ_α

Given a coloring $\mathcal{C}_P = \{X_i\}_{i=1}^m$ of P , the set $\{\varphi_\alpha(X_i)\}_{i=1}^m$ does not necessarily yield a coloring of Q_α . The problem is since φ_α is not one-to-one, it is possible that $\varphi_\alpha(X_i) \cap \varphi_\alpha(X_j) \neq \emptyset$ for some $i, j \in \{1, \dots, m\}$ and $i \neq j$. In that case, two points of P that are assigned different colors in \mathcal{C}_P have the same image under φ_α . If it happens that $\{\varphi_\alpha(X_i)\}_{i=1}^m$ is a coloring of Q_α , then we say that \mathcal{C}_P is compatible with φ_α . That is, each point of Q_α is assigned exactly one color when φ_α is applied on \mathcal{C}_P . We denote the coloring of Q_α obtained in this manner by $\varphi_\alpha(\mathcal{C}_P)$.

We now characterize the sublattice colorings of P that are compatible with φ_α .

Theorem 12. Let P be the set of centers of a regular tiling, and $\varphi_\alpha(z) = \exp(2\pi iz/\alpha)$ with α admissible. Then the coloring \mathcal{C}_P of P induced by a sublattice Γ is compatible with φ_α if and only if $\alpha \in \Gamma$.

Proof. Suppose \mathcal{C}_P is compatible with φ_α . Let $t \in P$. Since $\alpha \in \Lambda$, $t + \alpha \in P$. By Lemma 2, $\varphi_\alpha(t) = \varphi_\alpha(t + \alpha)$ and so t and $t + \alpha$ must belong to the same coset of Γ . Hence, $\alpha = (t + \alpha) - t \in \Gamma$.

Conversely, let $t, t' \in P$ such that $\varphi_\alpha(t) = \varphi_\alpha(t')$. This means that for some $k \in \mathbb{Z}$, $t' - t = k\alpha \in \Gamma$ since $\alpha \in \Gamma$. Thus, t' and t are in the same coset of Γ . \square

The following is immediate from Theorem 12.

Corollary 13. Let P be the set of centers of a regular tiling, \mathcal{C}_P the coloring of P induced by an ideal (β) and $\varphi_\alpha(z) =$

$\exp(2\pi iz/\alpha)$ with α admissible. Then \mathcal{C}_P is compatible with φ_α if and only if $(\alpha) \subseteq (\beta)$.

Hence, given φ_α , we can identify all ideal colorings of P that are compatible with φ_α by looking at the prime decomposition of α in $\mathbb{Z}[\xi]$, where $\xi = i$ if \mathcal{T} is the (4^4) tiling and $\xi = \omega$ if \mathcal{T} is the (6^3) or (3^6) tiling.

5.3. Perfect colorings of \mathcal{S}_α

Let \mathcal{C}_P be a coloring of P that is compatible with φ_α . We now give a necessary and sufficient condition for $g \in S(\mathcal{S}_\alpha)$ to be a color symmetry of the coloring $\varphi_\alpha(\mathcal{C}_P)$ of Q_α .

Theorem 14. Let P be the set of centers of a regular tiling, and $\varphi_\alpha(z) = \exp(2\pi iz/\alpha)$ with α admissible. Suppose a coloring \mathcal{C}_P of P is compatible with φ_α . If $g \in S(\mathcal{S}_\alpha)$ and $f \in S(\mathcal{T})$ corresponds to g , then g is a color symmetry of the coloring $\varphi_\alpha(\mathcal{C}_P)$ of Q_α if and only if f is a color symmetry of \mathcal{C}_P .

Proof. Suppose g is a color symmetry of $\varphi_\alpha(\mathcal{C}_P)$. If $X_i \in \mathcal{C}_P$, then $\varphi_\alpha(f(X_i)) = g(\varphi_\alpha(X_i)) = \varphi_\alpha(X_j)$ for some $X_j \in \mathcal{C}_P$. It follows from Lemma 2 that $f(X_i) = X_j + k\alpha$ for some $k \in \mathbb{Z}$. Since φ_α is compatible with \mathcal{C}_P and $\varphi_\alpha(X_j) = \varphi_\alpha(X_j + k\alpha)$, we have $f(X_i) = X_j$. Thus, f permutes the colors of \mathcal{C}_P .

In the other direction, let f be a color symmetry of \mathcal{C}_P . If $X_i \in \mathcal{C}_P$ then $g(\varphi_\alpha(X_i)) = \varphi_\alpha(f(X_i)) = \varphi_\alpha(X_j)$ for some $X_j \in \mathcal{C}_P$. Thus, g permutes the colors of $\varphi_\alpha(\mathcal{C}_P)$. \square

Theorem 14 provides a link between color symmetries of colorings of P and symmetries of \mathcal{S}_α that permute the colors of the corresponding coloring of Q_α . The next corollary follows from Theorem 1, Theorem 6 and Theorem 14.

Corollary 15. Let P be the set of centers of a regular tiling, $\varphi_\alpha(z) = \exp(2\pi iz/\alpha)$ with α admissible, and \mathcal{C}_P be the coloring of P induced by an ideal (β) that is compatible with φ_α .

(a) Then the elements of $R(\mathcal{S}_\alpha)$ are color symmetries of the coloring $\varphi_\alpha(\mathcal{C}_P)$ of Q_α .

(b) If α and β are balanced, then the elements of $S(\mathcal{S}_\alpha)$ are color symmetries of $\varphi_\alpha(\mathcal{C}_P)$.

We should note here that not all chirally and fully perfect colorings of \mathcal{S}_α arise from ideal colorings of P . In fact, every coloring of P induced by a sublattice Γ that contains α will yield a chirally perfect coloring of \mathcal{S}_α .

The next corollary confirms the observation by Lück (2010) that there is an upper bound for the number of colors in chirally perfect colorings of \mathcal{S}_α obtained from ideal colorings of P .

Corollary 16. Let P be the set of centers of a regular tiling and $\varphi_\alpha(z) = \exp(2\pi iz/\alpha)$ with α admissible. Suppose \mathcal{C}_P is an ideal coloring of P that is compatible with φ_α . Then there is an upper bound M for the number of colors in the coloring $\varphi_\alpha(\mathcal{C}_P)$ of Q_α . In particular:

(a) if \mathcal{T} is the (4^4) or (6^3) tiling, then $M = |\alpha|^2$;

(b) if \mathcal{T} is the (3^6) tiling, then $M = |\alpha|^2$ if $\alpha \notin P_0 = (2 + \omega)$ and $M = (2/3)|\alpha|^2$ if $\alpha \in P_0$.

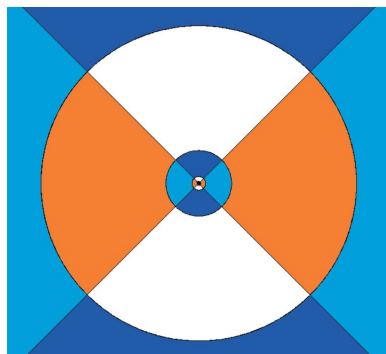


Figure 7
A fully perfect coloring of \mathcal{S}_α , with $\alpha = 4$, obtained from the coloring of P induced by (2).

Moreover, there exists a coloring of Q_α derived from some ideal coloring of P that has exactly M colors which are permuted by elements of $S(\mathcal{S}_\alpha)$.

Proof. It follows from Corollary 13 that (α) is the ideal that satisfies the compatibility condition and gives the most number of colors. Then $M = [\mathbb{Z}[\xi] : (\alpha)] = |\alpha|^2$ if \mathcal{T} is the (4^4) or (6^3) tiling. Lemma 9 gives us the value of M if \mathcal{T} is the (3^6) tiling. Finally, the elements of $S(\mathcal{S}_\alpha)$ permute the colors in $\varphi_\alpha(\mathcal{C}_P)$ obtained from \mathcal{C}_P induced by (α) by Corollary 15. \square

Note that the maximum number of colors in Corollary 16 is an inherent property of the tiling \mathcal{S}_α , and hence depends on the value of α . Thus, we can always find a singular tiling \mathcal{S}_α whose upper limit on the number of colors is as large as we wish by choosing a suitable admissible value of α .

However, unlike colorings of Q_α obtained from ideal colorings of P , there is no upper bound for the number of colors of colorings of Q_α obtained from sublattice colorings of P . For instance, if $\alpha = L + R\xi$ where $R \neq 0$, then the coloring \mathcal{C}_P of P induced by the sublattice Γ generated by α and k is compatible with φ_α for all $k \in \mathbb{Z}$ by Theorem 12. Observe that the number of colors in \mathcal{C}_P is given by $[\mathbb{Z}[\xi] : \Gamma] = |kR|$, which can be made arbitrarily large by the choice of k .

6. Summary and examples

Given an admissible α , we applied the conformal map φ_α on a regular tiling \mathcal{T} to come up with a tiling (\mathcal{S}_α) with a singularity at the origin. The symmetry group $S(\mathcal{S}_\alpha)$ is finite and either cyclic or dihedral, and a correspondence between the elements of $S(\mathcal{T})$ and $S(\mathcal{S}_\alpha)$ allowed us to identify $S(\mathcal{S}_\alpha)$.

Symmetric colorings of (\mathcal{S}_α) may be obtained by applying φ_α on compatible sublattice colorings \mathcal{C}_P of P . The compatibility condition requires that the sublattice that induces \mathcal{C}_P contains α . In particular, if the sublattice is the ideal (β) , then \mathcal{C}_P is compatible with φ_α exactly if $(\alpha) \subseteq (\beta)$. The coloring of \mathcal{S}_α derived from this compatible ideal coloring is chirally perfect. Furthermore, if α and β are both balanced, then the resulting coloring of \mathcal{S}_α is perfect.

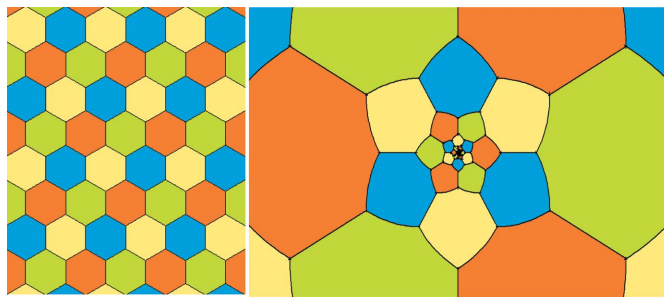


Figure 8
Perfect colorings of \mathcal{T} and \mathcal{S}_α , with $\alpha = 6$, induced by (2).

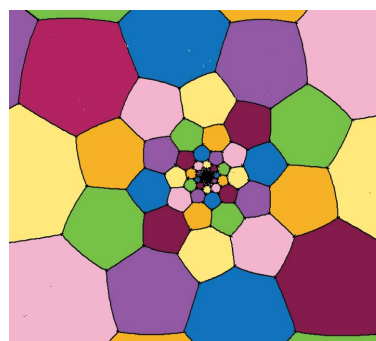


Figure 9
A perfect coloring of \mathcal{S}_α , with $\alpha = 10 + 2\omega$, obtained from a coloring of P induced by $(1 + 3\omega)$.

However, it is possible to obtain chirally (or fully) perfect colorings of \mathcal{S}_α from sublattice colorings of P that are not chirally (or fully) perfect. In fact, any sublattice coloring of P that is compatible with φ_α yields a chirally perfect coloring of \mathcal{S}_α .

Lastly, the maximum number of colors for a coloring of \mathcal{S}_α obtained from a compatible ideal coloring of P is $|\alpha|^2$ or $(2/3)|\alpha|^2$. On the other hand, no such constraint applies to colorings of \mathcal{S}_α derived from compatible sublattice colorings of P .

We end this paper with examples of chirally and fully perfect colorings of tilings with a singular point obtained from ideal colorings of lattices associated with regular tilings.

6.1. (4^4) Tiling

Let $\alpha = -5 + 5i$. Then \mathcal{S}_α is a Class 2 tiling and $S(\mathcal{S}_\alpha) \cong D_5$ is generated by $g_1(z) = \exp(2\pi i/5)z$ and $g_2(z) = \bar{z}$. The symmetries g_1 and g_2 of \mathcal{S}_α correspond to the symmetries $f_1(z) = z + (-1 + i)$ and $f_2(z) = iz$ of \mathcal{T} , respectively. Ideal colorings of $P = \mathbb{Z}[i]$ compatible with φ_α are induced by the ideals (1) , $(-1 + i)$, $(1 + 2i)$, $(1 - 2i)$, $(1 + 3i)$, $(3 + i)$, (5) and $(-5 + 5i)$. The colorings of \mathcal{S}_α that correspond to the colorings of P induced by the ideals (1) , $(-1 + i)$, (5) and $(-5 + 5i)$ are fully perfect. The maximum color index is 50. Fig. 1(a) shows a chirally perfect coloring of \mathcal{S}_α obtained from the coloring of P induced by $(2 + i)$.

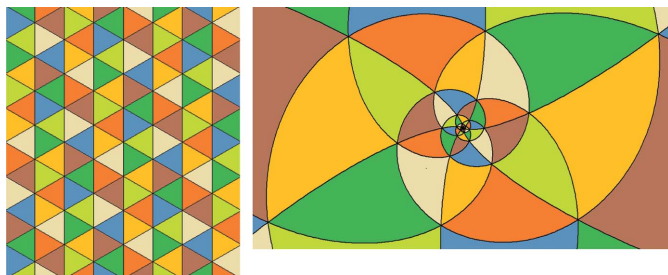


Figure 10
A perfect coloring of \mathcal{S}_α , with $\alpha = 2 + 10\omega$, obtained from a coloring of P induced by the ideal $(2 + 3\omega)$.

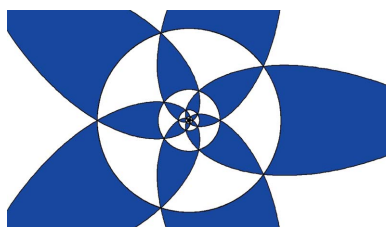


Figure 11
A fully perfect coloring of \mathcal{S}_α , with $\alpha = -5 + 5\omega$, obtained from a coloring of P induced by $(1 + 2\omega)$.

Fig. 7 shows a coloring of a Class 3 tiling \mathcal{S}_α , where $\alpha = 4$, obtained from the coloring of P induced by (2) . Since 2 is balanced, the coloring is fully perfect.

6.2. (6^3) Tiling

If $\alpha = 6$, then \mathcal{S}_α is a Class 3 tiling. The generators of $S(\mathcal{S}_\alpha) \cong D_6$ are $g_1(z) = \exp(\pi i/3)z$ and $g_2(z) = \bar{z}$, which correspond to the symmetries $f_1(z) = z + 1$ and $f_2(z) = -\bar{z}$ of \mathcal{T} , respectively. The ideals (1) , $(2 + \omega)$, (2) , (3) , $(2 + 4\omega)$ and (6) induce colorings of $P = \mathbb{Z}[\omega]$ that are compatible with φ_α . All resulting colorings of \mathcal{S}_α are perfect, and the maximum color index is 36. Fig. 8 shows the perfect colorings of \mathcal{T} and \mathcal{S}_α induced by (2) .

In Fig. 9, we have a coloring of a Class 1 tiling \mathcal{S}_α with $\alpha = 10 + 2\omega$. The coloring is perfect and is induced by the ideal $(1 + 3\omega)$.

6.3. (3^6) Tiling

We obtain a Class 1 tiling \mathcal{S}_α when $\alpha = 2 + 10\omega = (4 + 6\omega)(2 + \omega)$. The symmetry group of \mathcal{S}_α is isomorphic to C_2 , with generator $g_1(z) = -z$ that corresponds to the symmetry $f(z) = z + (1 + 5\omega)$ of \mathcal{T} . Colorings of $P = \mathbb{Z}[\omega] \setminus (2 + \omega)$ compatible with φ_α are induced by the ideals (1) , $(1 + 2\omega)$, (2) , $(2 + 3\omega)$, $(2 + 4\omega)$, $(1 + 5\omega)$, $(4 + 6\omega)$ and $(2 + 10\omega)$. The maximum color index is 56. Fig. 10 shows the chirally perfect coloring of \mathcal{T} and the fully perfect coloring of \mathcal{S}_α induced by $(2 + 3\omega)$.

Fig. 11 shows the fully perfect coloring of a Class 2 tiling \mathcal{S}_α , where $\alpha = -5 + 5\omega = 5\omega(2 + \omega)$, obtained from a coloring of P induced by $(1 + 2\omega)$.

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